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Some bounds on the injective chromatic number of graphs

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ABSTRACT

We give some bounds on the injective chromatic number.

1. Introduction, notation and terminology

The injective chromatic number of a graph was first defined as such in [3] even though it had been studied in several different guises and contexts before, albeit sometimes not in general; see [3] for more. Apart from various other considerations, the same paper also gives some general bounds and necessary and sufficient conditions for a regular graph to have the injective chromatic number equal to its degree. This paper looks at the problem under some conditions on the maximum average degree.

Our graphs are finite and simple (see [1] for undefined terms) and we will be colouring their vertices with nonnegative integers. Let $G = (V(G), E(G))$ be a graph. A *vertex colouring* (or, simply, a *colouring*) of G is a function $c : V(G) \longrightarrow \mathbb{N}$ (here, $0 \in \mathbb{N}$). A vertex k-colouring is a function $c : V(G) \longrightarrow [k]$, with $[k] = \{0, 1, \ldots, k-1\}$. We say that a colouring of a graph is *injective* if its restriction to the neighbourhood of any vertex is injective. The *injective chromatic number* $\chi_i(G)$ of a graph G is the least k such that there is an injective k-colouring. Clearly $\Delta(G) \leq \chi_i(G) \leq |V(G)|$ (as usual, $\Delta(G)$ is the maximum degree of a vertex of G). For (frequent) future reference, recall that, given a graph G and vertices $u, v \in V(G)$, the graphs $G - u$ and $G - uv$ are obtained from G, respectively, by removing u and all edges incident with it, or the edge uv, if it exists.

An obvious alternate way of looking at the injective chromatic number of a graph G is to consider the *common neighbour graph* $G^{(2)}$ of G defined by $V(G^{(2)}) = V(G)$ and $E(G^{(2)}) = \{[u, v]$: there is a path of length 2 in G joining u and $v\}$. Then $\chi_i(G) = \chi(G^{(2)})$. Other, related, colourings can be defined; again, see [3]. For easier comprehension consider the almost trivial results on the injective chromatic number of, for example, the complete graph, the path, the cycle and the star.

The *maximum average degree* is a well used tool and is defined as $\mathrm{MAD}(G) = \max\{\frac{2|E(H)|}{|V(H)|}$: H is a subgraph of $G\}$. The degree of a vertex u will be denoted by $\deg(u)$.

Note that the maximum average degree of a graph can be computed in polynomial time by using the Matroid Partitioning Algorithm due to Edmonds [2,5].

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Our main result in this paper is the following.

Theorem 1. (1) *If* G *is a graph with* $\mathrm{MAD}(G) < \frac{14}{5}$ *then* $\chi_i(G) \le \Delta(G) + 3$.
(2) *If* G *is a graph with* $\mathrm{MAD}(G) < 3$ *then* $\chi_i(G) \le \Delta(G) + 4$.
(3) *If* G *is a graph with* $\mathrm{MAD}(G) < \frac{10}{3}$ *then* $\chi_i(G) \le \Delta(G) + 8$.

2. Basic known results

Let us recall a few results and observations from [3]. We collect them in a lemma.

Lemma 2. *Let* G *be a graph.*
(1) *If* G *is* d*-regular and if* $\chi_i(G) = d$ *then* d *divides* $|V(G)|$.
(2) *If* G *is connected and not* K_2 *then* $\chi(G) \le \chi_i(G)$.
(3) *If* G *has diameter 2 and independence number* α *then* $\chi_i(G) \ge \alpha$.
(4) $\chi_i(G) \le \Delta(\Delta - 1) + 1$.
(5) *The bounds of* 3 *and* 4 *can be attained simultaneously.*

In the special case of the hypercube – which motivated [3] – we have slightly more, even though the value of $\chi_i(Q_n)$ is elusive in general.

Theorem 3. *Let* Q_n *be the hypercube of dimension* n.
(1) $\chi_i(Q_n) = n$ *if and only if* n *is a power of 2.*
(2) $\chi_i(Q_n) \le 2n - 2$.
(3) $\chi_i(Q_{2^m - j}) = 2^m$ *for* $0 \le j \le 3$.
(4) $\chi_i(Q_{2n+1}) \le 2\chi_i(Q_{n+1})$.

Surprising as it may seem, this is (almost) the extent of current knowledge of the parameter. The little extra that is known concerns a connection to linear codes, *NP*-completeness of the problem *Given* G *and an integer* k, *is* $\chi_i(G) \le k$? even if k is fixed, and some results on extremal graphs; see [3].

In this paper we give a few general upper bounds, in each case with an assumption on the maximum average degree of the graph.

3. Proof of the theorem

Let us first get rid of the simplest and uninteresting cases. Consider a graph G with maximum degree Δ. If $\Delta = 1$, G is a disjoint union of independent edges (a matching) and isolated vertices and so $\chi_i(G) = 1$. If $\Delta = 2$, G is a disjoint union of cycles, paths and isolated vertices and so $\chi_i(G) \le 3$. This is slightly less obvious but easy. A path $x_0 \dots x_{n-1}$ can be coloured by c defined by $c(x_i) = \lfloor \frac{i-1}{2} \rfloor (\mathrm{mod}\ 2)$; that is, starting with x_1, x_2, successive pairs are alternately coloured 0 or 1. A similar strategy works for cycles, but the number of colours will depend on the congruence modulo 4 of the number of vertices: $\chi_i(C_n) = 2$ if $n \equiv 0 (\mathrm{mod}\ 4)$, $\chi_i(C_n) = 3$ otherwise, as the reader can easily verify. We can, and will, therefore assume that $\Delta(G) \ge 3$.

We need a few more definitions and some notation. Let G be a graph and $u \in V(G)$. Define $N^2_G(u) = \cup_{v \in N(u)} N(v) \setminus \{u\}$ and $\deg^2_G(u) = |N^2_G(u)|$ (we will omit the subscript when it is clear from the context). Let $c : V(G) \longrightarrow [k]$ be a colouring of $V(G)$ and define $F^c_G(u) = \{c(x) : x \in N^2_G(u)\}$ and $C^c_G(u) = [k] \setminus F^c_G(u)$ (here we might also forget the subscript or the superscript if they are clear from the context). The colours in $F^c_G(u)$ are *forbidden* at u while those in $C^c_G(u)$ are *available* at u. Clearly $|F^c_G(u)| \le \deg^2(u)$.

Assume for the moment that G is triangle-free. Let now $uv \in E(G)$ and assume that $c : V(G) \longrightarrow [k]$ is an injective colouring of $G - uv$. Assume further that neither $C^c_G(u)$ nor $C^c_G(v)$ is empty (note that two graphs, G and $G - uv$, are at play here, with a common colouring c). Then $\hat{c} : V(G) \longrightarrow [k]$ defined by $\hat{c}(x) = c(x)$ if $x \notin \{u, v\}$ and $\hat{c}(x) \in C^c_G(x)$ if $x \in \{u, v\}$ is an injective colouring of G with the same number of colours. In particular, if c is a $\chi_i(G - uv)$-colouring of $G - uv$, $\chi_i(G) = \chi_i(G - uv)$. We call the pair (u, v) *recolourable with respect to* c if $\hat{c}$ can be defined as above; that is, if both $C^c_G(u)$ and $C^c_G(v)$ are non-empty and c is injective on $G - uv$. We call the pair (u, v) *recolourable* if it is recolourable with respect to any injective colouring of $G - uv$.
Consider now a graph G with triangles. Let u and v be adjacent vertices of G. Then $N(u) \cap N(v) \subseteq N^2(u), N^2(v)$ and so if c is an injective k-colouring of G as above, the colours forbidden at u include those of its neighbours common with v (similarly for v). In particular, $u \in N^2(v)$ and vice versa. Thus the above discussion applies equally to graphs with triangles.

A graph G is k*-critical* if $\chi_i(G) = k$ but $\chi_i(G - uv) < k$ for any edge uv of G and $\chi_i(G - u) < \chi_i(G)$ for any vertex u of v. When the value of k is not important, we will call the graph χ_i-critical. It is easy to see that if G is χ_i-critical then $\chi_i(G - uv) = \chi_i(G) - 1$ for any edge uv of G. Indeed, any injective colouring of $G - uv$ with $k - 1$ colours can be extended to an injective colouring of G with k colours simply by giving the vertices u and v the same new colour. On the other hand, for a vertex u of G, $\chi_i(G - u)$ and $\chi_i(G)$ can be arbitrarily far apart. Let $k \in \mathbb{N}$ and consider the graph G_k consisting of k triangles on vertex sets $\{u, u_i, v_i\}$, u a fixed common vertex, $i = 1, \dots, k$. Then $\chi_i(G_k) = 2k + 1$ while $\chi_i(G - u) = 1$.

It is easy to see that the only χ_i-critical graphs with $\Delta(G) = 2$ are the cycles of length $n \not\equiv 0 (\mathrm{mod}\ 4)$ and the path of length 3. This follows from the fact that, for a cycle C_n, $\chi_i(C_n) = 2$ if and only if $n \equiv 0 (\mathrm{mod}\ 4)$ and the fact that $\chi_i(P_n) = 2$

for $n > 2$, as mentioned previously. It is obvious that there are no χ_i-critical graphs with $\Delta(G) \leq 1$. We may, therefore, assume for the rest of the paper that $\chi_i(G) \geq 3$, as is also evident from the assumption that $\Delta(G) \geq 3$.

The following simple observations are some of those elevated to the status of lemmas because of their usefulness.

Lemma 4. *If G is χ_i-critical, it has no recolourable pairs.*

Lemma 5. *Let G be a χ_i-critical graph. Then $\chi_i(G) > \Delta(G)$.*

Proof. Suppose to the contrary that $\chi_i(G) = \Delta(G)$. Then, for any $u \in V(G)$, $\Delta(G - u) < \Delta(G)$. Thus every vertex of G is adjacent to all the vertices of maximum degree, which implies that G is a complete graph. But $\chi_i(K_n) = n > \Delta(K_n)$ for all $n > 1$ and, in particular, for $n > 3$. $\square$

This lemma suggests that we consider graphs G with $\chi_i(G) = \Delta(G) + t$, $t \in \mathbb{N} \setminus \{0\}$.

Lemma 6. *In a χ_i-critical graph the minimum degree is at least* 2.

Proof. Suppose that there is an edge uv in G with $\deg(u) = 1$. Then the pair (u, v) is recolourable. $\square$

Lemma 7. *Let G be a χ_i-critical graph and let $u \in V(G)$ be a vertex with* $\deg^2(u) < \chi_i(G) - 1$. *Then for every $v \in N(u)$,* $\deg^2(v) \geq \chi_i(G) - 1$.

Proof. If there is a vertex $v \in N(u)$ with $\deg^2(v) < \chi_i(G)$, the pair (u, v) is recolourable with respect to any injective colouring of $G - uv$. $\square$

Lemma 8. *Let $t \in \mathbb{N}$, $t > 0$, and let G be a $(\Delta(G) + t)$-critical graph.*

(1) *If uvz is a simple path in G with* $\deg(u) = 2$, $\deg(v) = 3$. *Then* $\deg(z) \geq t$.
(2) *A vertex of degree $d \leq t$ has at least two neighbours of degree at least* 3.

Proof. (1) If $\deg(z) \leq t$ then (u, v) is recolourable.
(2) Let u be a vertex with $\deg(u) \leq t + 1$ with only one neighbour, say w, of degree possibly greater than 2. Then for any $v \in N(u) \setminus \{w\}$, the pair (u, v) is recolourable. $\square$

From Lemma 8(2) we have the following simple corollary that will be used in the proofs of our theorems.

Corollary 9. *Let G be a $(\Delta(G) + t)$-critical graph, $t \geq 2$. Then both neighbours of a vertex of degree* 2 *have degree at least* 3.

The propositions that follow are all of the same form: *If* $\mathrm{MAD}(G) \leq x$ *then* $\chi_i(G) \leq \Delta(G) + t$ *with x a real and t a natural number*. The proofs also follow the same pattern: since a minimal counterexample is χ_i-critical, we show that if the conclusion does not hold for a χ_i-critical graph G, there is a discharging procedure that leads to a contradiction.

Discharging procedure. Let G be a graph and let $w : V(G) \longrightarrow \mathbb{R}$ be a weight function. Let $P, Q \subseteq V(G)$ be properties of vertices of G. A *discharging procedure* is a set of rules of the form "A vertex satisfying a property P gives $x_{PQ} \in \mathbb{R}$ to each vertex satisfying a property Q". We can thus define, for each vertex $u \in P$, the set $D(u) = N(u) \cap Q$ and set $D(u) = \emptyset$ if $u \notin P$. We then let $d(u, v) = x_{PQ}$ if $v \in D(u)$. With this, we can define $w' : V(G) \longrightarrow \mathbb{R}$ by

$$w'(u) = w(u) - \sum_{v \in D(u)} d(u, v) + \sum_{u \in D(v)} d(v, u)$$

and say that *the discharging procedure defines w'*.

The following observation will be used implicitly throughout.

Observation 1. *Let G be a graph and let uv be an edge of G. Let $G' = G - uv$ be the graph obtained from G by removing the edge uv from $E(G)$. Then* $\mathrm{MAD}(G') \leq \mathrm{MAD}(G)$ *and* $\chi_i(G') \leq \chi_i(G)$.

Lemma 10. *Let G be a $(\Delta(G) + 4)$-critical graph. It has the following properties.*

(1) *A vertex of degree* 4 *has at most two neighbours of degree* 2.
(2) *A vertex of degree* 4 *cannot have two neighbours of degree* 2 *and two neighbours of degree* 3.

Proof. Both follow easily from the proof of Lemma 8(2). $\square$

Now we prove the first part of Theorem 1:

Proposition 11. *Let G be a graph with* $\mathrm{MAD}(G) < \frac{14}{5}$ *then* $\chi_i(G) \leq \Delta(G) + 3$.

Proof. Consider a $(\Delta(G) + 4)$-critical graph G. Let $w : V(G) \longrightarrow \mathbb{R}$ be a charge function defined by $w(u) = \deg(u)$ for each $u \in V(G)$. We claim that the following discharging procedure defines w' such that $w'(u) \geq \frac{14}{5}$ for each $u \in V(G)$ and $\sum_{u \in V(G)} w(u) = \sum_{u \in V(G)} w'(u)$.

- If $uv \in E(G)$ and $\deg(u) = 2$, $\deg(v) \geq 3$, v gives $\frac{2}{5}$ to u.
- If $uv \in E(G)$ and $\deg(u) = 3$, $\deg(v) \geq 4$, v gives $\frac{1}{10}$ to u.

By Corollary 9, a vertex u of degree 2 has $w'(u) = 2 + 2\frac{2}{5} = \frac{14}{5}$. By Lemma 8(2), a vertex u of degree 3 has at most one neighbour of degree 2 and so $w'(u) \geq 3 - \frac{2}{5} + 2 \cdot \frac{1}{10} = \frac{14}{5}$. Let u be a vertex of degree 4. By Lemma 10(1) and (2), if u has two neighbours of degree 2, it has at most one neighbour of degree 3, and then $w'(u) \geq 4 - 2 \cdot \frac{2}{5} - \frac{1}{10} > \frac{14}{5}$. If u has fewer than two neighbours of degree 2, $w'(u) \geq 4 - 3 \cdot \frac{1}{10} > \frac{14}{5}$. Finally, if u is of degree at least 5 then $w'(u) \geq \deg(u) - \deg(u) \cdot \frac{2}{5} \geq \deg(u) \cdot \frac{3}{5} > 3$. Hence $w'(u) \geq \frac{14}{5}$ for every $u \in V(G)$ and so $\mathrm{MAD}(G) \geq \frac{14}{5}$.

We conclude that if G is a $(\Delta(G) + 4)$-critical graph, $\mathrm{MAD}(G) \geq \frac{14}{5}$. This proves the theorem. □

Lemma 12. *Let G be a $(\Delta(G) + 5)$-critical graph. It has the following properties.*

(1) *A vertex u of degree 3 with a neighbour v of degree 2 has the other neighbours of degree at least 5.*
(2) *A vertex of degree 4 has at most two neighbours of degree 2.*
(3) *A vertex of degree 4 with 2 neighbours of degree 2 cannot have a neighbour of degree 3.*
(4) *A vertex of degree 5 has at most four neighbours of degree 2.*
(5) *A vertex of degree 5 with four neighbours of degree 2 cannot have a neighbour of degree 3.*

Proof. (1) This follows easily from Lemma 8(1).
(2) If not, let $uv, uw, uz \in E(G)$ be such that $\deg(u) = 4, \deg(v) = \deg(w) = \deg(z) = 2$. Then the pair (u, v) is recolourable.
(3) If not, let $uv, uw, uz, \in E(G)$, $\deg(u) = 4$, $\deg(v) = \deg(w) = 2$ and $\deg(z) = 3$. Then (u, v) is a recolourable pair.
(4) Otherwise let $uv, uw, ux, uy, uz \in E(G)$ with $\deg(u) = 5, \deg(v) = \deg(w) = \deg(x) = \deg(y) = \deg(z) = 2$. The pair (u, v) is then recolourable.
(5) Otherwise let $uv, uw, ux, uy, uz \in E(G)$ with $\deg(u) = 5, \deg(v) = \deg(w) = \deg(x) = \deg(y) = 2$ and $\deg(z) = 3$. The pair (u, v) is then recolourable. □

Now we prove the second part of Theorem 1

Proposition 13. *Let G be a graph with $\mathrm{MAD}(G) < 3$ then $\chi_i(G) \leq \Delta(G) + 4$.*

Proof. Consider a $(\Delta(G) + 5)$-critical graph G. Let $w : V(G) \longrightarrow \mathbb{R}$ be a charge function defined by $w(u) = \deg(u)$ for each $u \in V(G)$. The discharging procedure we use to define w' is this.

- If $uv \in E(G)$ and $\deg(v) = 2$, $\deg(u) \geq 3$, u gives $\frac{1}{2}$ to v.
- If uvw is a simple path in G such that $\deg(u) \geq 4$, $\deg(v) = 3$, $\deg(w) = 2$, u gives $\frac{1}{4}$ to v.

We claim that $w'(u) \geq 3$ for each $u \in V(G)$.

- It follows from Corollary 9 that a vertex u with $\deg(u) = 2$ has $w'(u) = 2 + 2 \cdot \frac{1}{2} = 3$.
- From Lemma 12(1) we have that a vertex u of degree 3 has $w'(u) = w(u)$ for either it has a neighbour of degree 2 and then $w'(u) = 3 - \frac{1}{2} + 2 \cdot \frac{1}{4} = 3$, or it does not, in which case its weight does not decrease.
- From Lemma 12(2) and (3) we obtain that a vertex u of degree 4 has $w'(u) \geq 4 - 2 \cdot \frac{1}{2} = 3$ if it has a neighbour of degree 2 and $w'(u) \geq 4 - 4 \cdot \frac{1}{4} = 3$ if it has none.
- From Lemma 12(4) and (5) we get that if u has degree 5 then $w'(u) = 5 - 4 \cdot \frac{1}{2} = 3$ if it has four neighbours of degree 2 and $w'(u) \geq 5 - 3 \cdot \frac{1}{2} - 2 \cdot \frac{1}{4} = 3$ otherwise.
- Finally, for a vertex u of degree at least 6, $w'(u) \geq \deg(u) - \deg(u) \cdot \frac{1}{2} \geq 3$.

This proves the claim.

Thus $\mathrm{MAD}(G) \geq 3$ for a $(\Delta(G) + 5)$-critical graph and the proof of the theorem is complete. □

Lemma 14. *Let G be a $(\Delta(G) + 9)$-critical graph. It has the following properties.*

(1) *A vertex u of degree 3 with a neighbour v of degree 2 has the other neighbours of degree at least 9.*
(2) *At least one of any pair of adjacent vertices of degree 3 has both other neighbours of degree at least 8.*
(3) *A vertex of degree 4 with two neighbours of degree 2 has the other neighbours of degree at least 8.*
(4) *A vertex of degree 4 with one neighbour of degree 2 and one neighbour of degree 3 has the other neighbours of degree at least 7.*
(5) *A vertex of degree 5 with three neighbours of degree 2 has two neighbours of degree at least 7.*
(6) *A vertex of degree 6 with four neighbours of degree 2 has two neighbours of degree at least 6.*
(7) *A vertex of degree 7, 8 or 9 has at least two neighbours of degree at least 3.*

Proof. Proofs are similar to those of Lemma 12 and, as there, are done by simple counting and using Lemma 7. □

Now we prove the third part of Theorem 1

Proposition 15. *Let G be a graph with $\mathrm{MAD}(G) < \frac{10}{3}$ then $\chi_i(G) \leq \Delta(G) + 8$.*

Proof. Consider a $(\Delta(G)+9)$-critical graph G. As before, let $w : V(G) \longrightarrow \mathbb{R}$ be a charge function defined by $w(u) = \deg(u)$ for each $u \in V(G)$. The discharging procedure is as follows. Let u and v be two adjacent vertices of G.

(1) If $\deg(v) = 2$ and $\deg(u) \geq 3$, u gives $\frac{2}{3}$ to v.

(2) If $\deg(v) = 3$, $\deg(u) \geq 3$, and the other neighbours of u have degree at least 8, u gives $\frac{1}{9}$ to v.
(3) If $\deg(v) = 3$ and $\deg(u) \in \{4, 5, 6, 7\}$, u gives $\frac{1}{9}$ to v.
(4) If $\deg(v) = 4$ and $\deg(u) = 7$, u gives $\frac{1}{18}$ to v.
(5) If $\deg(v) = 5$ and $\deg(u) \geq 7$, u gives $\frac{1}{6}$ to v.
(6) If $\deg(v) = 3$ and $\deg(u) = 8$, u gives $\frac{2}{9}$ to v.
(7) If $\deg(v) = 4$ and $\deg(u) \geq 8$, u gives $\frac{1}{3}$ to v.
(8) If $\deg(v) = 3$ and $\deg(u) \geq 9$, u gives $\frac{1}{2}$ to v.

We now check that $w'(u) \geq \frac{10}{3}$ for all $u \in V(G)$. Understandably, we do not consider vertices that have degree at least 4 and do not give any charge to their neighbours for their new charge is clearly more than $\frac{10}{3}$.

(1) If u is of degree 2, it has only neighbours of degree 3 or more, by Corollary 9, and rule (1) applies. Thus $w'(u) = 2+2\cdot\frac{2}{3} = \frac{10}{3}$.
(2) If $\deg(u) = 3$, we have several cases.
If u has a neighbour of degree 2, it only has neighbours of degree at least 9, by Lemma 14(1). The rules 1 and 8 apply, and $w'(u) = 3 + 2 \cdot \frac{1}{2} - \frac{2}{3} = \frac{10}{3}$.
If a neighbour x of u is of degree 3, one of the two, say u, has the other neighbours of degree at least 8. Hence, with rule 6, $w'(u) = 3 + 2 \cdot \frac{2}{9} - \frac{1}{9} = \frac{10}{3}$ and, with rules 2 and $3 w'(x) = 3 + 3 \cdot \frac{1}{9} = \frac{10}{3}$.
If u only has neighbours of degree at least 4, $w'(u) \geq 3 + 3 \cdot \frac{1}{9} = \frac{10}{3}$, with rule 3.
(3) If $\deg(u) = 4$, it may have two neighbours of degree 2, in which case the remaining neighbours have degree at least 8, by Lemma 14(3). In this case rules 1 and 7 apply and $w'(u) = 4 + 2 \cdot \frac{1}{3} - 2 \cdot \frac{2}{3} = \frac{10}{3}$.
If u has a neighbour of degree 2 and a neighbour of degree 3, its remaining neighbours have degree at least 7, by Lemma 14(4), and so $w'(u) \geq 4 + 2 \cdot \frac{1}{18} - \frac{2}{3} - \frac{1}{9} = \frac{10}{3}$ using rules 4, 7, 1 and 2.
(4) If $\deg(u) = 5$ and u has three neighbours of degree 2, it has the remaining neighbours of degree at least 7, by Lemma 14(5). The rules 1 and 5 apply and $w'(u) = 5 + 2 \cdot \frac{1}{6} - 3 \cdot \frac{2}{3} = \frac{10}{3}$. In all other cases the charge remains at least $\frac{10}{3}$.
(5) If $\deg(u) = 6$ and if u has four neighbours of degree 2, it has the other neighbours of degree at least 6, by Lemma 14(6). Using rule 1, $w'(u) = 6 - 4 \cdot \frac{2}{3} = \frac{10}{3}$. In all other cases the charge remains at least $\frac{10}{3}$.
(6) If $\deg(u) = d \in \{7, 8, 9\}$ then it has at least two neighbours of degree at least 3, by Lemma 14(7). Thus $w'(u) \geq d - (d-2)\frac{2}{3} + 2 \cdot \frac{1}{9} \geq \frac{10}{3}$.

This concludes the proof. □

We can apply our results to planar graphs by using the following well known observation based on the Euler's formula.

Observation 2. *If G is a planar graph with girth g, then* $\mathrm{MAD}(G) < \frac{2g}{g-2}$.

Corollary 16. *Let G be a planar graph*

(1) *If $g(G) \geq 7$ then $\chi_i(G) \leq \Delta(G) + 3$.*
(2) *If $g(G) \geq 6$ then $\chi_i(G) \leq \Delta(G) + 4$.*
(3) *If $g(G) \geq 5$ then $\chi_i(G) \leq \Delta(G) + 8$.*

Lužar, Škrekovski, Tancer [4], having seen a preprint of this paper, improved the Corollary as follows.

Theorem 17. *Let G be a planar graph*

(1) *If $g(G) \geq 19$ then $\chi_i(G) = \Delta(G)$, $\Delta \geq 4$*
(2) *If $g(G) \geq 10$ then $\chi_i(G) \leq \Delta(G) + 1$, $\Delta \geq 4$*
(3) *If $g(G) \geq 5$ then $\chi_i(G) \leq \Delta(G) + 4$, $\Delta \geq 139$.*

Remark 18. The second author would like to point out two misprints in [3]. The first is the claim in page 4 that $\chi(G) = \chi_i(G)$ for the graph obtained from C_{6t} by adding the main diagonals. The correct example is C_{6t+1} for $t \equiv 1 \pmod 3$; there are others. The second error is the claim on page 5 that the diameter of the incidence graph of the projective plane is 2 (it is 3). This, however, does not falsify the rest of the observations.

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